**Exercise 1.** Let $X$ be a Banach space and assume that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence that weakly converges towards $x_\infty \in X$. Show that

$$\frac{1}{n}\sum_{k=1}^{n} x_i \underset{n\to\infty}{\rightharpoonup} x_\infty$$

too in the weak topology.

**Exercise 2.** Let $I = ]0,1[$, $\alpha \geq 0$, and $\{u_n\}_{n\in\mathbb{N}}$ be such that

$$u_n(x) = \begin{cases} n^\alpha & \text{for all } 0 \leq x \leq \dfrac{1}{n} \\ 0 & \text{for all } \dfrac{1}{n} < x < 1. \end{cases}$$

Show that if $1 < p < \infty$, we have

1. $u_n \underset{n\to\infty}{\longrightarrow} 0$ strongly in $L^p(I)$ if and only if $0 \leq \alpha < \dfrac{1}{p}$.
2. $u_n \underset{n\to\infty}{\rightharpoonup} 0$ weakly in $L^p(I)$ if and only if $0 \leq \alpha \leq \dfrac{1}{p}$.

**Exercise 3.** Let $I = ]0,2\pi[$ and $\{u_n\}_{n\in\mathbb{N}}$ be such that for all $n \in \mathbb{N}$, we have $u_n(x) = \sin(nx)$ for all $x \in ]0,2\pi[$. Show that:

1. For all $1 \leq p \leq \infty$, the sequence $\{u_n\}_{n\in\mathbb{N}}$ does not strongly converge towards 0 in $L^p(I)$.
2. For all $1 \leq p < \infty$, the sequence $\{u_n\}_{n\in\mathbb{N}}$ converge weakly towards 0 in $L^p(I)$.

**Exercise 4.** Let $\Omega \subset \mathbb{R}^d$ be an open set and assume that $1 \leq p < \infty$.

1. Let $\{u_n\}_{n\in\mathbb{N}} \subset L^p(\Omega)$ such that $u_n \underset{n\to\infty}{\rightharpoonup} u_\infty \in L^p(\Omega)$ weakly in $L^p(\Omega)$ and let $\{v_n\}_{n\in\mathbb{N}} \subset L^{p'}(\Omega)$ such that $v_n \underset{n\to\infty}{\longrightarrow} v_\infty \in L^{p'}(\Omega)$ strongly in $L^p(\Omega)$. Show that

$$u_n v_n \underset{n\to\infty}{\rightharpoonup} u_\infty v_\infty \text{ weakly in } L^1(\Omega).$$

   Find a counterexample showing that the result need not hold if we assume instead that $\{v_n\}_{n\in\mathbb{N}}$ converges weakly.

2. Show that if $\{u_n\}_{n\in\mathbb{N}} \subset L^2(\Omega)$ is such that $u_n \underset{n\to\infty}{\rightharpoonup} u_\infty \in L^2(\Omega)$ weakly in $L^2(\Omega)$ and $u_n^2 \underset{n\to\infty}{\rightharpoonup} u_\infty^2 \in L^1(\Omega)$ weakly in $L^1(\Omega)$, then we have the strong convergence

$$u_n \underset{n\to\infty}{\longrightarrow} u_\infty \qquad \text{in } L^2(\Omega).$$

**Exercise 5.** Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^d$ be a bounded set. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^p(\Omega)$ be a **bounded** sequence and assume that $f_n \underset{n\to\infty}{\longrightarrow} f$ almost everywhere.

1. Prove that $f_n \underset{n\to\infty}{\rightharpoonup} f$ weakly in $\sigma(L^p, L^{p'})$. You can admit that the weak limit and the limit almost everywhere coincide.
2. Now assume that $|\Omega| < \infty$. Show that for all $1 \leq q < p$, we have

$$\lim_{n\to\infty} \|f_n - f\|_{\mathrm{L}^p(\Omega)} = 0$$

**Exercise 6** (Brezis-Lieb Lemma). Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^d$ be a bounded set.

1. Prove that there exists a universal constant $C < \infty$ such that

$$||a+b|^p - |a|^p - |b|^p| \leq C\left(|a|^{p-1}|b| + |a||b|^{p-1}\right) \qquad \text{for all } \ a, b \in \mathbb{R}.$$

2. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^p(\Omega)$ be a **bounded** sequence and assume that $f_n \underset{n\to\infty}{\longrightarrow} f$ almost everywhere. Prove that $f \in L^p(\Omega)$ and that

$$\lim_{n\to\infty}\int_\Omega \left(|f_n|^p - |f_n - f|^p\right)dx = \int_\Omega |f|^p dx.$$

**Hint:** Use the previous question with $a = f_n - f$ and $b = f$ and use the previous exercise.

3. Deduce that if we assume furthermore that $\|f_n\|_{\mathrm{L}^p(\Omega)} \underset{n\to\infty}{\longrightarrow} \|f\|_{\mathrm{L}^p(\Omega)}$, then

$$\lim_{n\to\infty}\|f_n - f\|_{\mathrm{L}^p(\Omega)} = 0.$$